Solutions to short-answer questions

1 a

$$0.0\dot{7} = 0.07777\ldots$$
$$0.0\dot{7} \times 10 = 0.7777\ldots$$
$$0.0\dot{7} \times 9 = 0.7 = \frac{7}{10}$$
$$0.0\dot{7} = \frac{7}{90}$$

b

$$0.\dot{4}\dot{5} = 0.454545\ldots$$
$$0.\dot{4}\dot{5} \times 100 = 45.4545\ldots$$
$$0.\dot{4}\dot{5} \times 99 = 45$$
$$0.\dot{4}\dot{5} = \frac{45}{99} = \frac{5}{11}$$

c $0.005 = \frac{5}{1000} = \frac{1}{200}$

d $0.405 = \frac{405}{1000} = \frac{81}{200}$

e

$$0.2\dot{6} = 0.26666\ldots$$
$$0.2\dot{6} \times 10 = 2.6666\ldots$$
$$0.2\dot{6} \times 9 = 2.4 = \frac{24}{10}$$
$$0.2\dot{6} = \frac{24}{90} = \frac{4}{15}$$

f

$$0.1\dot{7}1428\dot{5}$$
$$= 0.1714825714\ldots$$
$$0.1\dot{7}1428\dot{5} \times 10^6$$
$$= 171\,428.5714285\ldots$$
$$0.1\dot{7}1428\dot{5} \times (10^6 - 1)$$
$$= 171\,428.4$$
$$= \frac{1\,714\,284}{10}$$
$$0.1\dot{7}1428\dot{5}$$
$$= \frac{1\,714\,284}{9\,999\,990}$$
$$= \frac{6}{35}$$

2

$$\begin{array}{r|l} 2 & 504 \\ 2 & 252 \\ 2 & 126 \\ 2 & 63 \\ 3 & 21 \\ 7 & 7 \\ & 1 \end{array}$$

$$504 = 2^3 \times 3^2 \times 7$$

3 a $|n^2 - 9|$ is prime.

$|n^2 - 9| = |n - 3||n + 3|$

For it to be prime either $|n - 3| = 1$ or $|n + 3| = 1$

If $|n - 3| = 1$, then $n = 4$ or $n = 2$

If $|n + 3| = 1$, then $n = -4$ or $n = -2$

b **i** $x^2+5|x|-6=0$

Consider two cases:

$x \geq 0: \quad x^2+5x-6=0$

$(x+6)(x-1)=0$

$\therefore x=1$

$x<0: \quad x^2-5x-6=0\ (x-6)(x+1)=0$

$\therefore x=-1$

ii $x+|x|=0$

Consider two cases:

$x \geq 0: 2x=0 \Rightarrow x=0$

$x<0$:Always true

Therefore the solution is $x \leq 0$

c $5-|x|<4$

$|x|>1$

$\therefore x>1$ or $x<-1$.

4 **a**
$$\begin{aligned}\frac{2\sqrt{3}-1}{\sqrt{2}} &= \frac{2\sqrt{3}-1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{2\sqrt{6}-\sqrt{2}}{2}\end{aligned}$$

b
$$\begin{aligned}\frac{\sqrt{5}+2}{\sqrt{5}-2} &= \frac{\sqrt{5}+2}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2} \\ &= \frac{5+4\sqrt{5}+4}{5-4} \\ &= 4\sqrt{5}+9\end{aligned}$$

c
$$\begin{aligned}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} &= \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} \\ &= \frac{3+2\sqrt{6}+2}{3-2} \\ &= 2\sqrt{6}+5\end{aligned}$$

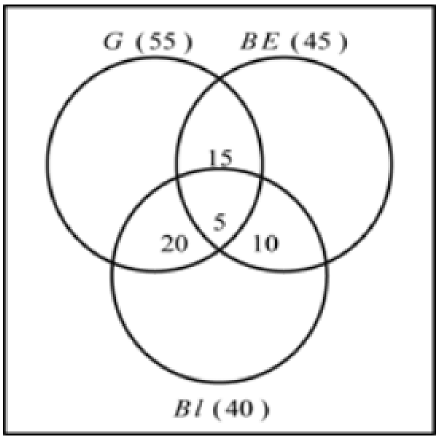
5
$$\begin{aligned}\frac{3+2\sqrt{75}}{3-\sqrt{12}} &= \frac{3+2\sqrt{25\times 3}}{3-\sqrt{4\times 3}} \\ &= \frac{3+2\times 5\sqrt{3}}{3-2\sqrt{3}} \\ &= \frac{3+10\sqrt{3}}{3-2\sqrt{3}} \times \frac{3+2\sqrt{3}}{3+2\sqrt{3}} \\ &= \frac{9+6\sqrt{3}+30\sqrt{3}+60}{9-12} \\ &= \frac{69+36\sqrt{3}}{-3} \\ &= -23-12\sqrt{3}\end{aligned}$$

6 **a**
$$\begin{aligned}\frac{6\sqrt{2}}{3\sqrt{2}-2\sqrt{3}} &= \frac{6\sqrt{2}}{3\sqrt{2}-2\sqrt{3}} \times \frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} \\ &= \frac{36+12\sqrt{6}}{18-12} \\ &= \frac{36+12\sqrt{6}}{6} \\ &= 6+2\sqrt{6}\end{aligned}$$

b
$$\frac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}} = \frac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}} \times \frac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}$$

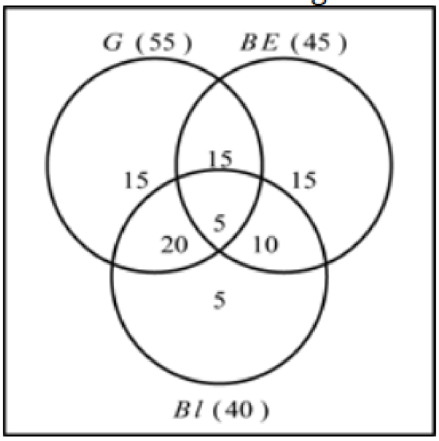
$$= \frac{a+b-2\sqrt{(a+b)(a-b)}+a-b}{(a+b)-(a-b)}$$
$$= \frac{2a-2\sqrt{a^2-b^2}}{2b}$$
$$= \frac{a-\sqrt{a^2-b^2}}{b}$$

7 First enter the information on a Venn diagram.



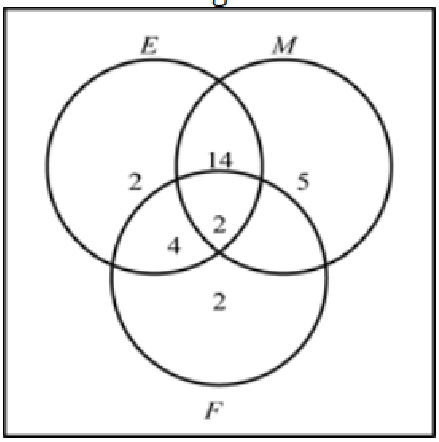
a It is obvious to make up the 40 blonds that 5 must be blond only, so the number of boys (not girls) who are blond is $5+10=15$.

b The rest of the Venn diagram can be filled in the same way:



Boys not blond or blue-eyed
$= 100 - 15 - 15 - 15 - 20 - 5 - 10 - 5$
$= 15$

8 Fill in a Venn diagram.

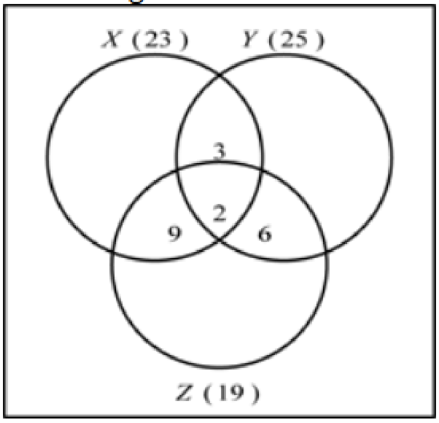


a $30-2-14-5-4-2-2=1$ (since all received at least one prize.)

b $14+5+2+1=22$

c $2+14+4+2=22$

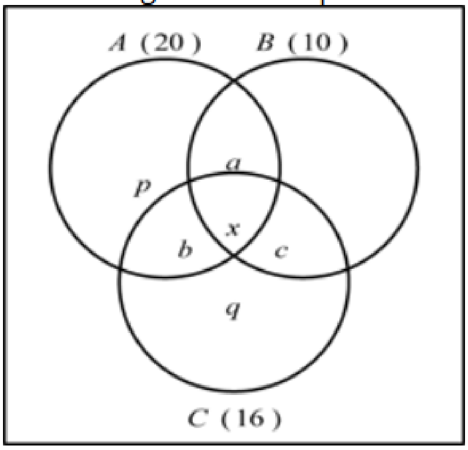
9 Enter the given information on a Venn diagram as below.



The numbers liking X only, Y only and Z only are 9, 14 and 2 respectively.

$$\begin{aligned}&\text{The number who like none of them}\\&=50-9-3-14-9-2-6-2\\&=5\end{aligned}$$

10 The rectangles can be represented by circles for clarity. Enter the data:



Note: $a+x=3$, $b+x=6$ and $c+x=4$

$$\begin{aligned}p+b+a+x&=20\\p+b+3&=20\\p+b&=17\\q+(p+b)+n(B)&=35\\q+17+10&=35\\\therefore q&=8\\q+(b+x)+c=n(C)&=16\\8+6+c&=16\\\therefore c&=2\\c+x&=4\\\therefore x&=2\end{aligned}$$

There is $2\ \text{cm}^2$ in common.

11 $$\begin{aligned}\sqrt{112}-\sqrt{63}-\frac{224}{\sqrt{28}}&=\sqrt{16\times7}-\sqrt{9\times7}-\frac{224}{\sqrt{4\times7}}\\&=4\sqrt{7}-3\sqrt{7}-\frac{224}{2\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}}\\&=4\sqrt{7}-3\sqrt{7}-\frac{224\sqrt{7}}{14}\\&=4\sqrt{7}-3\sqrt{7}-16\sqrt{7}\\&=-15\sqrt{7}\end{aligned}$$

12 Cross multiply:

$$\begin{aligned}(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3}) &= x^2\\ 7-3 &= x^2\\ 4 &= x^2\\ x &= \pm 2\end{aligned}$$

13

$$\begin{aligned}\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}+\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}} &= \frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}+\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}\\ &= \frac{\sqrt{5}-\sqrt{5}+\sqrt{10}-\sqrt{6}}{5-3}+\frac{\sqrt{5}+\sqrt{5}-\sqrt{10}-\sqrt{6}}{5-3}\\ &= \frac{2\sqrt{5}-2\sqrt{6}}{2}\\ &= \sqrt{5}-\sqrt{6}\end{aligned}$$

14

$$\begin{aligned}\sqrt{27}-\sqrt{12}+2\sqrt{75}-\sqrt{\frac{48}{25}} &= \sqrt{9\times 3}-\sqrt{4\times 3}+2\sqrt{25\times 3}-\frac{\sqrt{16\times 3}}{\sqrt{25}}\\ &= 3\sqrt{3}-2\sqrt{3}+10\sqrt{3}-\frac{4\sqrt{3}}{5}\\ &= \frac{15\sqrt{3}-10\sqrt{3}+50\sqrt{3}-4\sqrt{3}}{5}\\ &= \frac{51\sqrt{3}}{5}\end{aligned}$$

15a $|A\cup B| = 32+7+15+3 = 57$

b $C = 3$

c $B'\cap A = 32$

16

$$\begin{aligned}17+6\sqrt{8} &= 17+2\times\sqrt{9}\times\sqrt{8}\\ &= 17+2\sqrt{72}\end{aligned}$$

$a+b=17; ab=72$

$a=8,\ b=9$ (or $a=9,\ b=8$, giving the same answer.)

$(\sqrt{8}+\sqrt{9})^2 = 17+6\sqrt{8}$

So the square root of

$$\begin{aligned}17+6\sqrt{8} &= \sqrt{8}+\sqrt{9}\\ &= 2\sqrt{2}+3\end{aligned}$$

17

$$\begin{aligned}1885 &= 365\times 5+60\\ (1885,365) &= (60,365)\\ 365 &= 60\times 6+5\\ (60,365) &= (60,5)\\ 60 &= 5\times 12+0\\ (1885,365) &= 5\end{aligned}$$

18a Apply the division algorithm to 43 and 9.

$$\begin{aligned}43 &= 9\times 4+7\\ 9 &= 7\times 1+2\\ 7 &= 2\times 3+1\\ 2 &= 2\times 1\end{aligned}$$

Working backwards with these results,

$$\begin{aligned}1 &= 7-2\times 3\\ 1 &= 7-(9-7\times 1)\times 3\\ 1 &= 7-9\times 3+7\times 3\\ 1 &= 7\times 4-9\times 3\end{aligned}$$

$1 = (43 - 9 \times 4) \times 4 - 9 \times 3$
$1 = 43 \times 4 - 9 \times 16 - 9 \times 3$
$1 = 43 \times 4 - 9 \times 19$

A solution to $9x + 43y = 1$ is $x = -19, y = 4$.

A solution to $9x + 43y = 7$ is $x = -19 \times 7 = -133, y = 4 \times 7 = 28$.

The general solution is

$x = -133 + 43t$
$y = 28 - 9t, t \in \mathbb{Z}$

Other solutions are possible.

$t = 4$ gives a specific solution of $x = 39$,

$y = -8$, leading to a general solution of

$x = 39 + 43t$
$y = -8 - 9t,\ t \in \mathbb{Z}$

b If $x > 0, 39 + 43t > 0$

$$t > -\frac{39}{43}$$

If $y > 0, -8 - 9t > 0$

$$t < -\frac{8}{9}$$

These two inequations cannot both be true if x is an integer.

There is no solution for $x \in \mathbb{Z}^+,\ y \in \mathbb{Z}^+$.

19 If a and b are odd, they may be written as $2n + 1$ and $2m + 1$ respectively, where n and m are integers.

$$\begin{aligned} ab &= (2n+1)(2m+1) \\ &= 4mn + 2n + 2m + 1 \\ &= 2(2mn + n + m) + 1 \end{aligned}$$

This will be an odd number since $2mn + n + m$ is an integer.

20

$$\begin{aligned} 12\,121 &= 10\,659 \times 1 + 1462 \\ (12\,121, 10\,659) &= (1462, 10\,659) \\ 10\,659 &= 1462 \times 7 + 425 \\ (1462, 10\,659) &= (1462, 425) \\ 1462 &= 425 \times 3 + 187 \\ (1462, 425) &= (187, 425) \\ 425 &= 187 \times 2 + 51 \\ (187, 425) &= (187, 51) \\ 187 &= 51 \times 3 + 34 \\ (187, 51) &= (51, 34) \\ 51 &= 34 \times 1 + 17 \\ (51, 34) &= (34, 17) \\ 34 &= 17 \times 2 + 0 \\ (12\,121, 10\,659) &= 17 \end{aligned}$$

21a The algorithm is still useful.

$7 = 5 \times 1 + 2$
$5 = 2 \times 2 + 1$
$1 = 5 - 2 \times 2$
$1 = 5 - (7 - 5 \times 1) \times 2$
$1 = 5 - 7 \times 2 + 5 \times 2$
$1 = 5 \times 3 - 7 \times 2$

A solution is $x = 3, y = -2$.

The general solution is

$$x = 3 + 7t$$
$$y = -2 - 5t,\ t \in R$$

b If $1 = 5 \times 3 - 7 \times 2$, then $100 = 5 \times 300 - 7 \times 200$.

A solution is $x = 300, y = -200$.

The general solution is

$$x = 300 + 7t$$
$$y = -200 - 5t,\ t \in \mathbb{Z}$$
$$x = 300 + 7t, y = -200 - 5t$$

c If $y \geq x$,

$$-2 - 5t \geq 3 + 7t$$
$$-12t \geq 5$$
$$t \leq -\frac{5}{12}$$

Since t is an integer, $t \leq -1$.

The solution is

$$x = 3 + 7t$$
$$y = -2 - 5t, t \leq -1, t \in \mathbb{Z}$$

22 First, let Tom's age be t and Fred's age be f.

Since it appears Tom is older than Fred, and we must look at the time when Tom was Fred's age, we will define d as the difference in ages, specifically how many years older Tom is than Fred.

$$t = f + d$$
$$t + f = 63$$
$$\therefore\quad (f + d) + f = 63$$
$$2f + d = 63$$

When Tom was Fred's age, d years ago, Fred was aged $f - d$.

Tom is now twice that age, $2(f - d)$.

$$\therefore\quad t = 2(f - d)$$
$$\therefore\quad t = 2(f - d)$$

Since $t = f + d$,

$$f + d = 2(f - d)$$
$$= 2f - 2d$$
$$3d = f$$

Substitute $f = 3d$ into $2f + d = 63$.

$$6d + d = 63$$
$$7d = 63$$
$$d = 9$$
$$f = 3d$$
$$= 27$$
$$t + f = 63$$
$$t = 36$$

Tom is 36 and Fred is 27.

Solutions to multiple-choice questions

1 **A** $\dfrac{4}{3+2\sqrt{2}} = \dfrac{4}{3+2\sqrt{2}} \times \dfrac{3-2\sqrt{2}}{3-2\sqrt{2}}$

$$= \frac{12-8\sqrt{2}}{9-8}$$

$$= 12-8\sqrt{2}$$

2 **D**

$$\begin{array}{r|r} 2 & 86\,400 \\ 2 & 43\,200 \\ 2 & 21\,600 \\ 2 & 10\,800 \\ 2 & 5400 \\ 2 & 2700 \\ 2 & 1350 \\ 3 & 675 \\ 3 & 225 \\ 3 & 75 \\ 5 & 25 \\ 5 & 5 \\ \hline & 1 \end{array}$$

$\text{Prime decomposition} = 2^7 \times 3^3 \times 5^2$

3 **D** $(\sqrt{6}+3)(\sqrt{6}-3) = (\sqrt{6})^2 + 3\sqrt{6} - 3\sqrt{6} - 9$

$$= 6-9$$

$$= -3$$

4 **D** $B' \cap A = \text{numbers in set } A \text{that are not also in set } B$

$$= \{1, 2, 4, 5, 7, 8\}$$

5 **C** $(3, \infty) \cap (-\infty, 5] = \{x \in R : x > 3\} \cap \{x \in R : x \leq 5\}$

$$= \{x \in R : 3 < x \leq 5\}$$

$$= (3, 5]$$

6 **D** The next time will be both a multiple of 6 and a multiple of 14.

$$\text{LCM} = \frac{6 \times 14}{3}$$

$$= 42$$

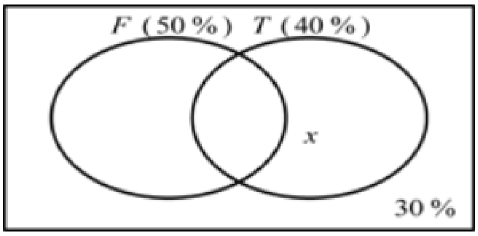
The next time is in 42 minutes.

7 **B** $X \cap Y \cap \mathbb{Z} = \text{set of numbers that are multiples of } 2, 5 \text{ and } 7$

$$\text{LCM} = 2 \times 5 \times 7$$

$$= 35$$

8 **B** Draw a Venn diagram.



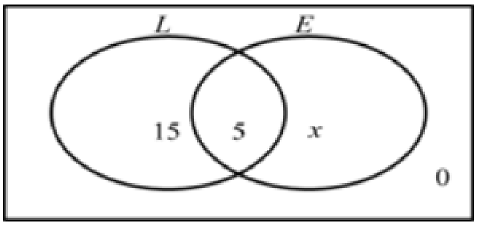
Since 50 don't play football,

$$\begin{aligned} x + 30\% &= 50\% \\ x &= 20\% \end{aligned}$$

Since 40% play tennis, it can be seen that 20% play both sports.

9 **C** $$\begin{aligned} \frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}} &= \frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}} \times \frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}-\sqrt{6}} \\ &= \frac{7-2\sqrt{42}+6}{7-6} \\ &= 13-2\sqrt{42} \end{aligned}$$

10 **A** Draw a Venn diagram.



$$\begin{aligned} 15+5+x &= 40 \\ x &= 20 \end{aligned}$$

20 students take only Economics.

11 **D**

12 **D** You can choose any number of 2s from 0 to p in $(p+1)$ ways. For each of these, you can choose any number of 3s from 0 to q in $(q+1)$ ways, and for each of these combinations you can choose any number of 5s from 0 to r in $(r+1)$ ways.

The total number of ways $= (p+1)(q+1)(r+1)$

13 **B** $$\begin{aligned} m+n &= mn \\ n &= mn-m \\ &= m(n-1) \\ m &= \frac{n}{n-1} \end{aligned}$$

This will only be an integer if $n=2, m=2$ or $n=0, m=0$.

There are two solutions.

Solutions to extended-response questions

1 **a** $$\begin{aligned} (\sqrt{x}+\sqrt{y})^2 &= (\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{y}) \\ &= \sqrt{x}(\sqrt{x}+\sqrt{y})+\sqrt{y}(\sqrt{x}+\sqrt{y}) \\ &= x+\sqrt{x}\sqrt{y}+\sqrt{y}\sqrt{x}+y \\ &= x+y+2\sqrt{x}\sqrt{y} \\ &= x+y+2\sqrt{xy} \end{aligned}$$

b From **a**, $$\begin{aligned} (\sqrt{3}+\sqrt{5})^2 &= 3+5+2\sqrt{3}\sqrt{5} \\ &= 8+2\sqrt{15} \\ \therefore \quad \sqrt{3}+\sqrt{5} &= \sqrt{8+2\sqrt{15}} \end{aligned}$$

c **i** $$\begin{aligned} (\sqrt{11}+\sqrt{3})^2 &= 11+3+2\sqrt{11}\sqrt{3} \\ &= 14+2\sqrt{33} \\ \therefore \sqrt{14+2\sqrt{33}} &= \sqrt{11}+\sqrt{3} \end{aligned}$$

ii $$\begin{aligned} (\sqrt{8}-\sqrt{7})^2 &= 8+7-2\sqrt{8}\sqrt{7} \text{ (also consider } -\sqrt{8}+\sqrt{7}) \\ &= 15-2\sqrt{56} \end{aligned}$$

$$\therefore \sqrt{15-2\sqrt{56}} = \sqrt{8}-\sqrt{7}$$
$$= 2\sqrt{2}-\sqrt{7}$$

iii $$(\sqrt{27}-\sqrt{24})^2 = 27+24-2\sqrt{27}\sqrt{24}$$
$$= 51-2\times 3\sqrt{3}\times 2\sqrt{3}\sqrt{2}$$
$$= 51-36\sqrt{2}$$
$$\therefore \sqrt{51-36\sqrt{2}} = \sqrt{27}-\sqrt{24}$$
$$= 3\sqrt{3}-2\sqrt{6}$$

2 a $$(2+3\sqrt{3})+(4+2\sqrt{3}) = 2+4+3\sqrt{3}+2\sqrt{3}$$
$$= 6+5\sqrt{3}$$

Hence $a = 6$ and $b = 5$.

b $$(2+3\sqrt{3})(4+2\sqrt{3}) = 2(4+2\sqrt{3})+3\sqrt{3}(4+2\sqrt{3})$$
$$= 8+4\sqrt{3}+12\sqrt{3}+18$$
$$= 26+16\sqrt{3}$$

Hence $p = 26$ and $q = 16$.

c $$\frac{1}{3+2\sqrt{3}} = \frac{1}{3+2\sqrt{3}}\times\frac{3-2\sqrt{3}}{3-2\sqrt{3}}$$
$$= \frac{3-2\sqrt{3}}{9-12}$$
$$= \frac{3-2\sqrt{3}}{-3}$$
$$= -1+\frac{2}{3}\sqrt{3}$$

Hence $a = -1$ and $b = \frac{2}{3}$.

d i $$(2+5\sqrt{3})x = 2-\sqrt{3}$$
$$\therefore x = \frac{2-\sqrt{3}}{2+5\sqrt{3}}$$
$$= \frac{2-\sqrt{3}}{2+5\sqrt{3}}\times\frac{2-5\sqrt{3}}{2-5\sqrt{3}}$$
$$= \frac{(2-\sqrt{3})(2-5\sqrt{3})}{4-75}$$
$$= \frac{2(2-5\sqrt{3})-\sqrt{3}(2-5\sqrt{3})}{-71}$$
$$= \frac{4-10\sqrt{3}-2\sqrt{3}+15}{-71}$$
$$= \frac{19-12\sqrt{3}}{-71}$$
$$= \frac{12\sqrt{3}-19}{71}$$

ii $$(x-3)^2-3 = 0$$
$$\therefore (x-3)^2 = 3$$
$$\therefore x-3 = \pm\sqrt{3}$$
$$\therefore x = 3\pm\sqrt{3}$$

iii $(2x-1)^2-3=0$

$$\therefore (2x-1)^2=3$$
$$\therefore 2x-1=\pm\sqrt{3}$$
$$\therefore 2x=1\pm\sqrt{3}$$
$$\therefore x=\frac{1\pm\sqrt{3}}{2}$$

e If $b=0,\ a+b\sqrt{3}=a$. Hence every rational number, a, is a member of $\{a+b\sqrt{3}: a,\ b\in\mathbb{Q}\}$.

3 a
$$\frac{1}{2+\sqrt{3}}=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}$$
$$=\frac{2-\sqrt{3}}{4-3}$$
$$=2-\sqrt{3}$$

b
$$(\sqrt{2+\sqrt{3}})^x+(\sqrt{2-\sqrt{3}})^x=4$$
$$\therefore (\sqrt{2+\sqrt{3}})^x+\left(\sqrt{\frac{1}{2+\sqrt{3}}}\right)^x=4\ \left(\text{as } \frac{1}{2+\sqrt{3}}=2-\sqrt{3}\text{ from } \mathbf{a}\right)$$
$$\therefore (\sqrt{2+\sqrt{3}})^x+\frac{1}{(\sqrt{2+\sqrt{3}})^x}=4$$
$$\therefore t+\frac{1}{t}=4\quad\text{where } t=(\sqrt{2+\sqrt{3}})^x$$

c
$$t+\frac{1}{t}=4$$
$$\therefore t^2+1=4t$$
$$\therefore t^2-4t+1=0$$

Using the general quadratic formula $t=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ with $a=1,\ b=-4$ and $c=1$ gives

$$t=\frac{4\pm\sqrt{(-4)^2-4\times1\times1}}{2\times1}$$
$$=\frac{4\pm\sqrt{16-4}}{2}$$
$$=\frac{4\pm\sqrt{12}}{2}$$
$$=\frac{4\pm2\sqrt{3}}{2}$$
$$=2\pm\sqrt{3}$$

d From **c**, $t=2+\sqrt{3}$ or $t=2-\sqrt{3}$
but $t=(\sqrt{2+\sqrt{3}})^x$,

$$\therefore (\sqrt{2+\sqrt{3}})^x=2+\sqrt{3}\qquad (1)$$
$$\text{or } (\sqrt{2+\sqrt{3}})^x=2-\sqrt{3}\qquad (2)$$

$$\text{From (1) } (2+\sqrt{3})^{\frac{x}{2}}=2+\sqrt{3}$$
$$\therefore \frac{x}{2}=1$$
$$\therefore x=2$$

$$\text{and from (2) } (2+\sqrt{3})^{\frac{x}{2}}=\frac{1}{2+\sqrt{3}}\quad \left(\text{as } \frac{1}{2+\sqrt{3}}=2-\sqrt{3}\text{ from } \mathbf{a}\right)$$
$$=(2+\sqrt{3})^{-1}$$

$$\therefore \frac{x}{2} = -1$$
$$\therefore x = -2$$

Hence the solutions of the equation $(\sqrt{2+\sqrt{3}})^x + (\sqrt{2-\sqrt{3}})^x = 4$ are $x = \pm 2$.

Graphics calculator techniques for Question 3

CAS calculator techniques for Question 3

3 d A CAS calculator can be used to help understand the structure of this question.
TI: Sketch the graphs of

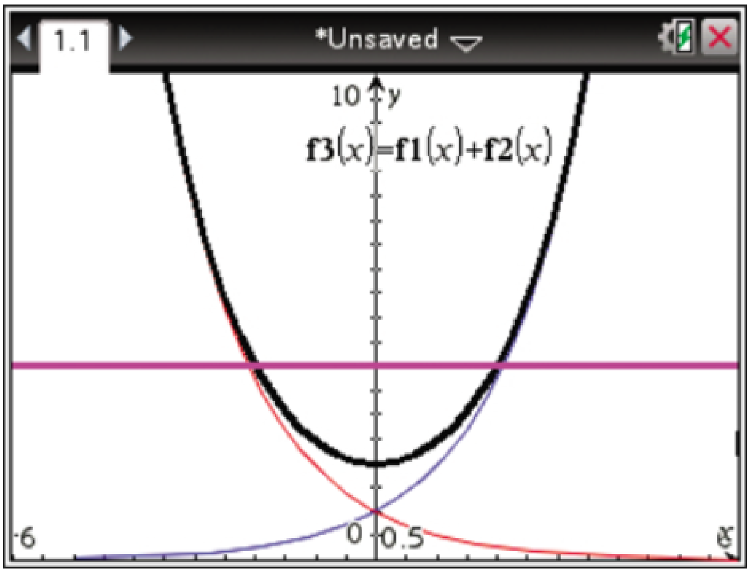
$\boldsymbol{f1} = \left(\sqrt{2+\sqrt{3}}\right)^x$,

$\boldsymbol{f2} = \left(\sqrt{2-\sqrt{3}}\right)^x$, $\boldsymbol{f3 = f1(x) + f2(x)}$ and

$\boldsymbol{f4 = 4}$

Press**Menu**→**6:Analyze Graph**→**4:Intersection**

Repeat this process to find the other intersection point



Alternatively, with the graphs still active, type $\mathbf{solve(f3(x) = 4, x)}$ in the Calculator application

CP: Sketch the graphs of

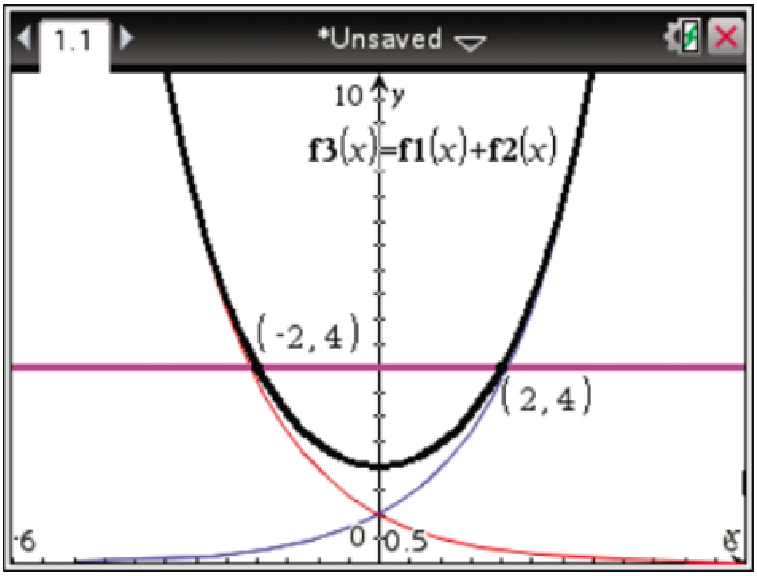
$\boldsymbol{y1} = \left(\sqrt{2+\sqrt{3}}\right)^x$,

$\boldsymbol{y2} = \left(\sqrt{2-\sqrt{3}}\right)^x$,

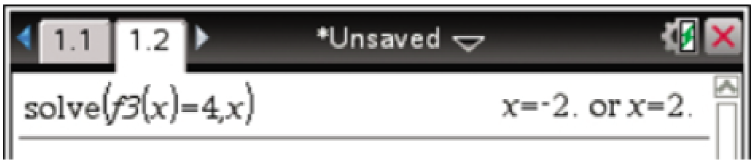
$\boldsymbol{y3 = y1(x) + y2(x)}$ and
$\boldsymbol{y4 = 4}$

Tap**Analysis**→**G-Solve**→**Intersect**

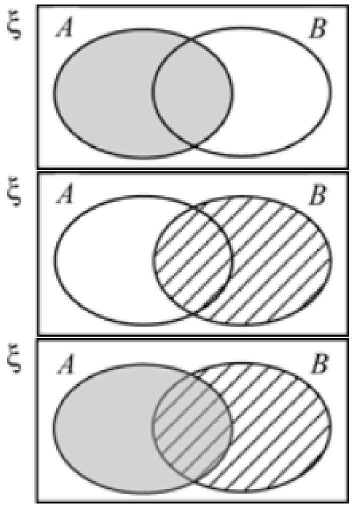
Use the Up and Down arrows on the Keypad to select the graph of y 3 and y 4



To display the other point of intersection use the Left and Right arrows

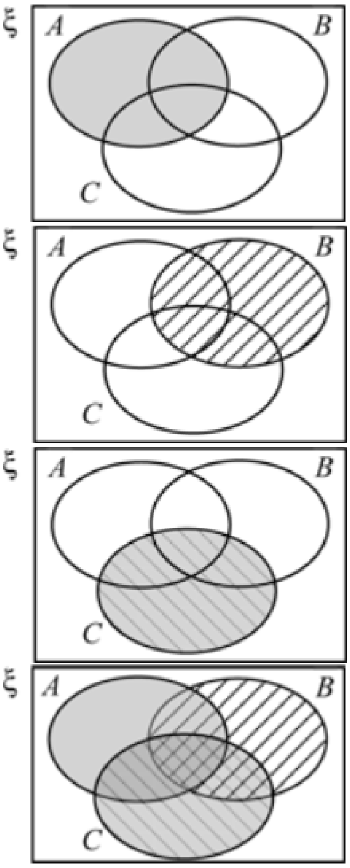


4 a



$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

b



$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

5 a If $x^2 + bx + c = 0$ and $x = 2 - \sqrt{3}$

$$\begin{aligned}
\text{then} \quad & (2-\sqrt{3})^2 + b(2-\sqrt{3}) + c = 0 \\
\therefore \quad & 4 - 4\sqrt{3} + 3 + 2b - \sqrt{3}b + c = 0 \\
\therefore \quad & (7 + 2b + c) + (-4 - b)\sqrt{3} = 0 \\
\therefore \quad & 7 + 2b + c = 0 \quad \text{and} \quad -4 - b = 0 \\
\therefore \quad & 7 + 2(-4) + c = 0 \quad b = -4 \\
\therefore \quad & 7 - 8 + c = 0 \\
\therefore \quad & -1 + c = 0 \\
\therefore \quad & c = 1
\end{aligned}$$

b $x^2 - 4x + 1 = 0$

Using the same procedure as in**3c**, $x = 2 \pm \sqrt{3}$.

Hence $2 + \sqrt{3}$ is the other solution.

c i If $x^2 + bx + c = 0$ and $x = m - n\sqrt{q}$

$$\begin{aligned}
\text{then} \quad & (m - n\sqrt{q})^2 + b(m - n\sqrt{q}) + c = 0 \\
\therefore \quad & m^2 - 2mn\sqrt{q} + n^2q + bm - bn\sqrt{q} + c = 0 \\
\therefore \quad & (m^2 + n^2q + bm + c) + (-2mn - bn)\sqrt{q} = 0 \\
\therefore \quad & m^2 + n^2q + bm + c = 0 \text{ and } -2mn - bn = 0 \\
& -2mn = bn \\
& -2m = b
\end{aligned}$$

ii

$$\begin{aligned}
& m^2 + n^2q + (-2m)m + c = 0 \\
\therefore \quad & m^2 + n^2q + -2m^2 + c = 0 \\
\therefore \quad & n^2q - m^2 + c = 0 \\
\therefore \quad & c = m^2 - n^2q
\end{aligned}$$

iii If $x^2+bx+c=0$, the general quadratic formula gives

$$x=\frac{-b\pm\sqrt{b^2-4c}}{2}\ (\text{as } a=1)$$

Given $b=-2m$ and $c=m^2-n^2q$

$$\begin{aligned}x&=\frac{2m\pm\sqrt{4m^2-4(m^2-n^2q)}}{2}\\&=\frac{2m\pm\sqrt{4m^2-4m^2+4n^2q}}{2}\\&=\frac{2m\pm 2n\sqrt{q}}{2}\\&=m\pm n\sqrt{q}\end{aligned}$$

$$\therefore\quad x^2+bx+c=(x-(m-n\sqrt{q}))(x-(m+n\sqrt{q}))$$

or, by completing the square,

$$\begin{aligned}x^2-2mx+m^2-n^2q&=x^2-2mx+m^2+m^2-n^2q-m^2\\&=(x-m)^2-(n\sqrt{q})^2\\&=(x-m-n\sqrt{q})(x-m+n\sqrt{q})\end{aligned}$$

6 a

$$\begin{aligned}x&=2mn\\&=2\times5\times2\\&=20\\y&=m^2-n^2\\&=5^2-2^2\\&=25-4\\&=21\\z&=m^2+n^2\\&=5^2+2^2\\&=25+4\\&=29\end{aligned}$$

b

$$\begin{aligned}x^2+y^2&=(2mn)^2+(m^2-n^2)^2\\&=4m^2n^2+m^4-2m^2n^2+n^4\\&=2m^2n^2+m^4+n^4\\z^2&=(m^2+n^2)^2\\&=m^4+2m^2n^2+n^4\end{aligned}$$

$\therefore x^2+y^2=z^2$

7 a i $2^3=8$. Factors of 8 are $1,2,4$ and 8. Hence 2^3 has four factors.

ii $3^7=2187$. Factors of 2187 are $1,3,9,27,81,243,729$ and 2187. Hence 3^7 has eight factors.

b

$2^1=2$ Factors are $1,2$. Hence 2^1 has two factors.
$2^2=4$ Factors are $1,2,4$. Hence 2^2 has three factors.
$2^3=8$ Factors are $1,2,4,8$. Hence 2^3 has four factors.
$2^4=16$ Factors are $1,2,4,8,16$. Hence 2^4 has five factors.
2^n has $n+1$ factors.

c **i** $2^1.3^1 = 6$. Factors are $1, 2, 3, 6$. There are four factors.

$2^1.3^2 = 18$. Factors are $1, 2, 3, 6, 9, 18$. There are six factors.

$2^2.3^2 = 36$. Factors are $1, 2, 3, 4, 6, 9, 12, 18, 36$. There are nine factors.

$2^2.3^3 = 108$. Factors are $1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108$. There are twelve factors.

$2^3.3^7$ has $(3+1)(7+1) = 32$ factors.

ii $2^n.3^m$ has $(n+1)(m+1)$ factors.

d The following table investigates the relationship between the number of factors of x and its prime factorisation.

x	Factors	Number of factors	Prime factorisation	Number of factors
1	1	1		$0+1$
2	$1, 2$	2	2^1	$1+1$
3	$1, 3$	2	3^1	$1+1$
4	$1, 2, 4$	3	2^2	$2+1$
5	$1, 5$	2	5^1	$1+1$
6	$1, 2, 3, 6$	4	$2^1.3^1$	$(1+1)(1+1)$

For any number x, there are $(\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\ldots(\alpha_n+1)$ factors.

e

$$\begin{aligned} 8 &= 4 \times 2 \\ &= (3+1)(1+1) \end{aligned}$$

$$\text{Now } 2^3.3^1 = 24$$

The smallest number which has eight factors is 24.

8 **a** $1080 = 2^3 \times 3^3 \times 5 \quad 25\,200 = 2^4 \times 3^2 \times 5^2 \times 7$

b Least common multiple of 1080 and $25\,200$ is $2^4 \times 3^3 \times 5^2 \times 7 = 75\,600$

c HCF of m and $n = p_1^{\min(\alpha_1, \beta_1)} p_2^{\min(\alpha_2, \beta_2)} \ldots p_n^{\min(\alpha_n, \beta_n)}$

$\therefore$ the product of the HCF and LCM

$$\begin{aligned} &= p_1^{\min(\alpha_1, \beta_1)+\max(\alpha_1, \beta_1)}\ p_2^{\min(\alpha_2, \beta_2)+\max(\alpha_2, \beta_2)} \ldots\ Pn^{\min(\alpha_n, \beta_n)+\max(\alpha_n, \beta_n)} \\ &= p_1^{\alpha_1+\beta_1} p_2^{\alpha_2+\beta_2}\ p_n^{\alpha_n+\beta_n} \\ &= mn \end{aligned}$$

d **i** The lowest common multiple of $5, 7, 9$ and 11 is 3465.

Now $3465 + 11$ is divisible by 11, $3465 + 9$ is divisible by 9, $3465 + 7$ is divisible by 7, $3465 + 5$ is divisible by 5.

Therefore choose numbers $3476, 3474, 3472$ and 3470.

ii Divide by 2 to obtain 4 consecutive natural numbers, i.e. $1738, 1737, 1736, 1735$.

9 **a** **i** Region 8, $B' \cap F' \cap R'$

ii Region 1, $B \cap F' \cap R$ represents red haired, blue eyed males.

iii Region 2, $B \cap F' \cap R'$ represents blue eyed males who do not have red hair.

b Let ξ be the set of all students at Argos Secondary College studying French, Greek or Japanese.

$n(\xi) = n(F \cup G \cup J) = 250$

$n(F' \cap G' \cap J') = 0$

$n((G \cup J) \cap F') = 41$

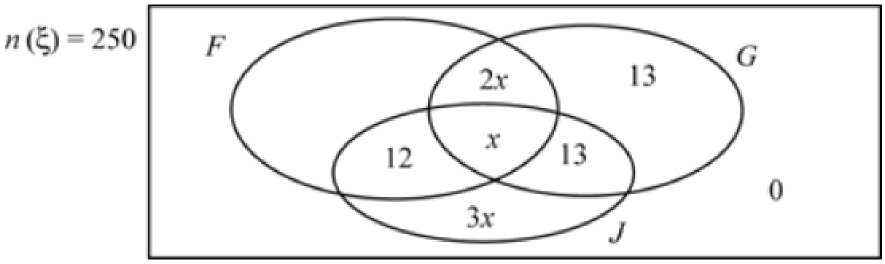
$n(F \cap J \cap G') = 12$

$n(J \cap G \cap F') = 13$

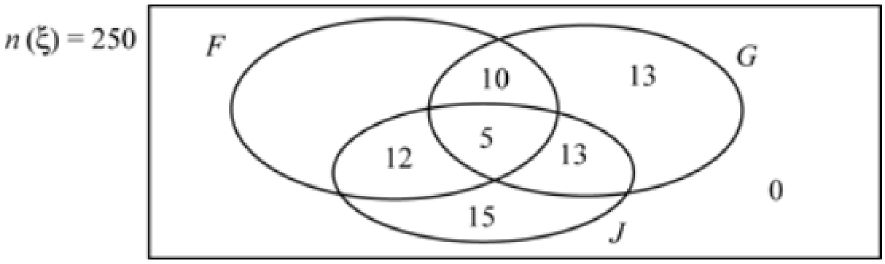
$n(G \cap J' \cap F') = 13$

$n(F \cap G \cap J') = 2 \times n(F \cap G \cap J)$

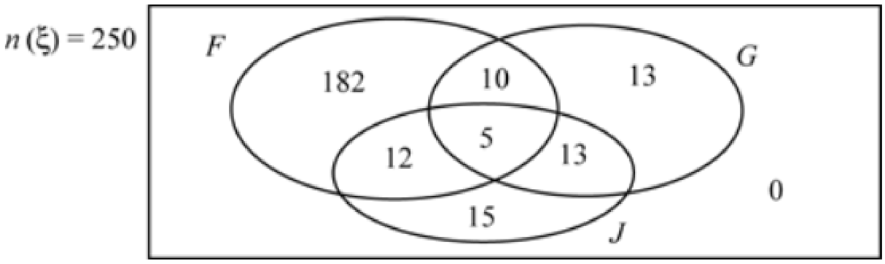
$n(J \cap G' \cap F') = n(F \cap G)$



$$
\begin{aligned}
\text{Now } n((G \cup J) \cap F') &= 13 + 13 + 3x \\
&= 26 + 3x \\
\therefore \quad 26 + 3x &= 41 \\
\therefore \quad 3x &= 15 \\
\therefore \quad x &= 5
\end{aligned}
$$



$$
\begin{aligned}
n(F \cap G' \cap J') &= 250 - (10 + 12 + 5 + 13 + 13 + 15 + 0) \\
&= 250 - 68 \\
&= 182
\end{aligned}
$$



i $n(F \cap G \cap J) = 5$, the number studying all three languages.

ii $n(F \cap G' \cap J') = 182$, the number studying only French.

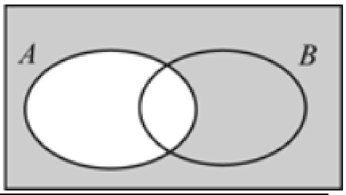
10a i B' denotes the set of students at Sounion Secondary College 180 cm or shorter.

ii $A \cup B$ denotes the set of students at Sounion Secondary College either femaleor taller than 180 cm or both.

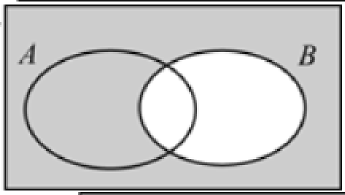
iii $A' \cap B'$ denotes the set of students at Sounion Secondary College who are males180 cm or shorter.

b

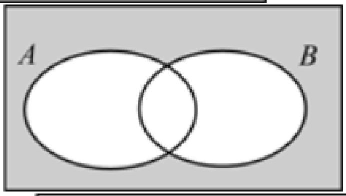
A'



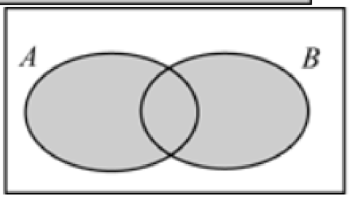
B'



$A' \cap B'$



$(A' \cap B')'$

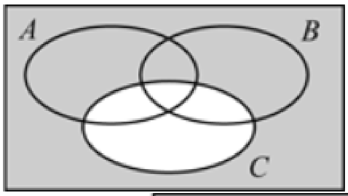


$A \cup B$

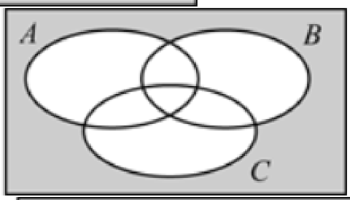
$\therefore A \cup B = (A' \cap B')'$

c

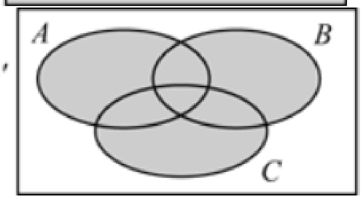
C'



$A' \cap B' \cap C'$



$(A' \cap B' \cap C')'$



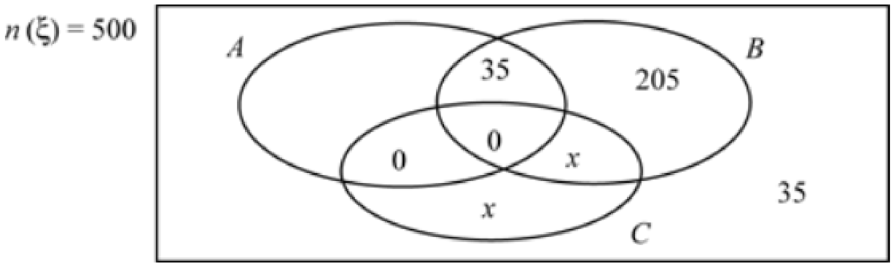
$A \cup B \cup C$

$\therefore A \cup B \cup C = (A' \cap B' \cap C')'$

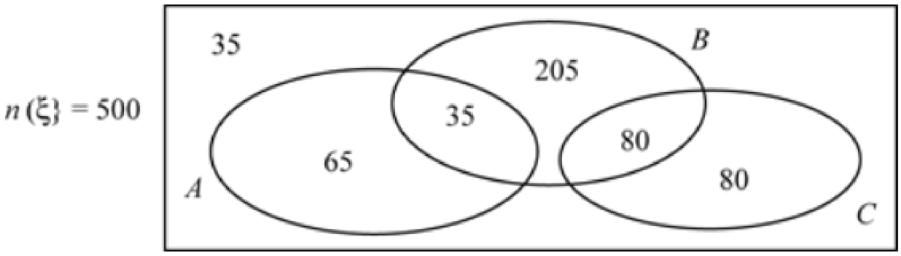
11

$$\begin{aligned} n(\xi) &= 500 \\ n(A \cap C) &= 0 \\ n(A) &= 100 \\ n(B \cap A' \cap C') &= 205 \\ n(C) &= 2 \times n(B \cap C) \\ n(A \cap B \cap C') &= 35 \\ n(A' \cap B' \cap C') &= 35 \end{aligned}$$

a



$$n(A \cap B' \cap C') = 100 - 35$$
$$= 65$$
$$2x + 35 + 65 + 205 + 35 = 500$$
$$\therefore \quad 2x + 340 = 500$$
$$\therefore \quad 2x = 160$$
$$\therefore \quad x = 80$$



b $n(C) = 160$, regular readers of C.

c $n(A \cap B' \cap C') = 65$, regular readers of A only.

d $n(A \cap B \cap C) = 0$, regular readers of A, B and C.